Although the suction-analogy, intuitively derived by Polhamus, may lack the aesthetics of theoretical rigor, its accuracy in describing complex vortex-lift effects has been systematically verified in Ref. 1 for both subsonic and supersonic speeds. The straightforward extension of the analogy to rectangular wings by Lamar and to complex planforms as presented here lends further credence to the utility of the concept as an aerodynamic analysis tool.


### References

[1]Polhamus, E. C., "Prediction of Vortex-Lift Characteristics by a Leading-Edge Suction Analogy," *Journal of Aircraft*, Vol. 8, No. 4, April 1971, pp. 193–199.

[2]Polhamus, E. C., "A Concept of the Vortex Lift of Sharp-Edge Delta Wings Based on a Leading-Edge-Suction Analogy," TN D-3767, Dec. 1966, NASA.

[3]Winter, H., "Flow Phenomena on Plates and Airfoils of Short Span," TM 798, July 1936, NACA.

[4]Jones, G. W. Jr., "Investigation of the Effects of Variations in the Reynolds Number Between 0.4 × 10⁶ and 3.0 × 10⁶ on the Low-Speed Aerodynamic Characteristics of Three Low-Aspect-Ratio Symmetrical Wings With Rectangular Planforms," RM L52G18, Sept. 1952, NACA.

[5]Nelson, W. H. and McDevitt, J. B., "The Transonic Characteristics of 22 Rectangular, Symmetrical Wing Models of Varying Aspect Ratio and Thickness," TN 3501, June 1955, NACA.

[6]Emerson, H. F., "Wind-Tunnel Investigation of the Effect of Clipping the Tips of Triangular Wings of Different Thickness, Camber, and Aspect Ratio—Transonic Bump Method," TN 3671, June 1956, NACA.

[7]Wentz, W. H. Jr. and Kohlman, D. L., "Wind Tunnel Investigations of Vortex Breakdown on Slender Sharp-Edged Wings," CR 98737, Nov. 1968, NASA.


# A Modified Wall Wake Velocity Profile for Turbulent Compressible Boundary Layers


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### Symbols

$A$ = $\{[(\gamma-1)/2]M_e^2/(T_w/T_e)\}^{1/2}$
$a$ = a constant, see Eq. (8)
$B$ = $\{(1+[(\gamma-1)/2]M_e^2)/(T_w/T_e)\}-1$
$C$ = constant in Law of the Wall (usually equals 5.1)
$C_f$ = skin friction coefficient $\tau_w/(1/2)\,\rho_e u_e^2$
$K$ = constant in mixing length (usually equals 0.4)
$M$ = Mach number
$Re_\delta$ = Reynolds number based on $\delta$
$u$ = velocity in streamwise direction
$u^*$ = $(u_e/A)\arcsin\{[(2A^2\,u/u_e)-B]/(B^2+4A^2)^{1/2}\}$
$u_\tau$ = Friction velocity $(\tau_w/\rho_w)^{1/2}$
$W$ = Coles' universal wake function
$y$ = coordinate normal to wall
$\gamma$ = ratio of specific heats
$\delta$ = boundary-layer thickness
$\eta$ = $y/\delta$
$\nu$ = kinematic viscosity
$\Pi$ = coefficient of wake function
$\rho$ = density
$\sigma$ = $[(\gamma-1)/2]M_e^2/\{1+[(\gamma-1)/2]M_e^2\}$
$\tau$ = shear stress

***Subscripts***

$e$ = conditions at the edge of the boundary layer
$w$ = conditions at the wall


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A SIMPLE representation of the mean velocity distribution in a turbulent boundary layer is very useful in integral analyses of turbulent flow problems. After an extensive survey of mean velocity profile measurements, Coles[1] suggested that for incompressible turbulent boundary layer flow the velocity profile may be represented by a linear combination of two universal functions in the form

$$u/u_\tau = (1/K)\ln(yu_\tau/\delta) + C + \Pi W(y/\delta)/K = f(y) + g(y) \tag{1}$$

where

$$f(y) = (1/K)\ln(yu_\tau/\delta) + C \tag{2}$$

is the Law of the Wall and

$$g(y) = \Pi(y/\delta)/K \tag{3}$$

is the Law of the Wake.

Setting $u/u_e$ and $W(y/\delta) = 2$ at $(y/\delta) = 1$ in Eq. (1), and subtracting the resulting equation from Eq. (1) leads to an expression for the velocity of the form

$$u/u_e = 1 + (1/K)(u_\tau/u_e)\ln(y/\delta) - (\Pi/K)(u_\tau/u_e)[(2 - W(y/\delta)] \tag{4}$$

Mathews et al.[2] have developed a wall-wake representation of the velocity profile in a form applicable for isoenergetic compressible boundary layers. Their profile is

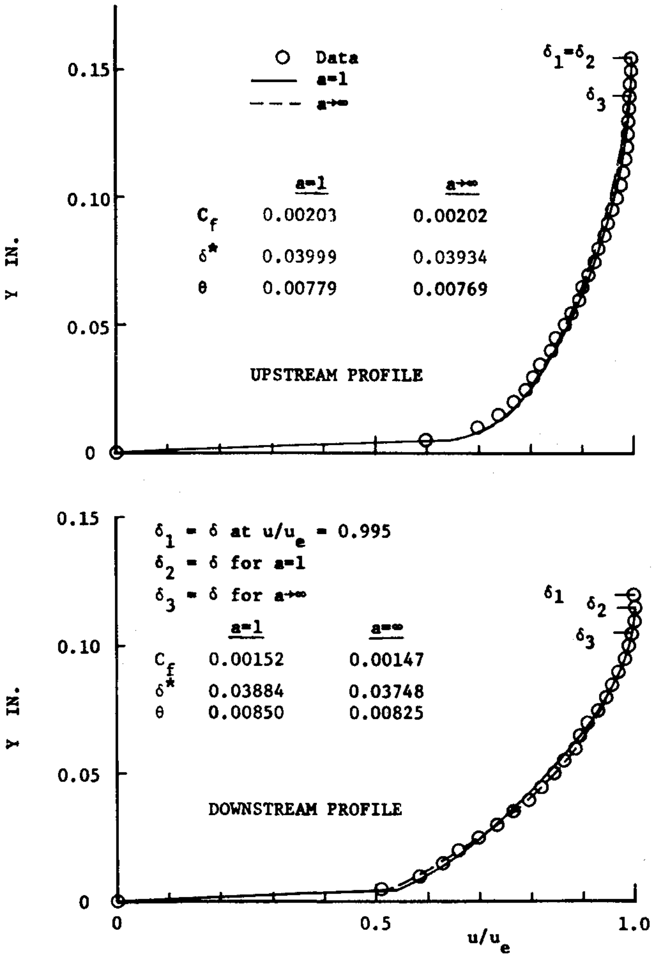


**Fig. 1 Velocity profiles upstream and downstream of a shock wave-boundary layer interaction.**

expressed as

$$u/u_e = (1/\sigma^{1/2}) \sin\{\text{arc}\sin\sigma^{1/2}[1 + (1/K)(u_\tau/u_e^*) \ln(y/\delta) - (\Pi/K)(u_\tau/u_e^*)(1 + \cos(\pi y/\delta))]\} \quad (5)$$

where

$$(u_\tau/u_e^*) = [(C_f/2)\sigma/(1-\sigma)]^{1/2}/\text{arc}\sin\sigma^{1/2} \quad (6)$$

and

$$\Pi/K = (1/2)\{[1/(u_\tau/u_e^*)] - (1/K)\ln[Re_\delta(C_f/2)^{1/2}(1-\sigma)^{1.26}] - C\} \quad (7)$$

and where $(2 - W)$ has been replaced by $1 + \cos(\pi y/\delta)$ for mathematical convenience.

Equation (5) has been found to provide a good representation of the boundary layer velocity profile for a range of external Mach numbers and wall static pressure gradients. However, with both Eq. (4) and Eq. (5) the velocity gradient at the boundary layer edge is found to have a nonzero value. In the modified wall-wake which is to be developed here this shortcoming is avoided.

The law of the wall may be derived from Prandtl's mixing length theory and the assumption that the shear stress is constant across the boundary layer (cf. Schlichting[3]). However, an expression for $\tau$ of the form

$$\tau = \tau_w[1 - (y/\delta)^a] = \tau_w(1 - \eta^a) \quad (8)$$

where $a$ is a real constant should provide a more realistic relationship for the shear stress.

Using Eq. (8) we may write

$$\tau_w(1 - \eta^a) = \rho K^2 y^2 (du/dy)^2 \quad (9)$$

Integration of Eq. (9) gives an expression for $u/u_\tau$ of the form

$$u/u_\tau = (1/K)\ln\eta + (2/aK)\{(1-\eta^a)^{1/2} - \ln[1 + (1-\eta^a)^{1/2}]\} + C_1 \quad (10)$$

Replacing $f(y)$ in Eq. (1) by Eq. (10) we have

$$u/u_\tau = (1/K)\ln\eta + (2/aK)\{(1-\eta^a)^{1/2} - \ln[1 + (1-\eta^a)^{1/2}]\} + C_1 + \frac{\Pi}{K}W(\eta) \quad (11)$$

At the boundary layer edge ($\eta \to 1$) we have

$$u_e/u_\tau = C_1 + (\Pi/K)W(1) = C_1 + 2\Pi/K \quad (12)$$

while near the wall (as $\eta \to 0^+$),

$$u/u_\tau = (1/K)\ln(yu_\tau/\nu) - (1/K)\ln(\delta u_\tau/\nu) + C_1 + 0.614/aK \quad (13)$$

Near the wall the expression for the law of the wall as given by Eq. (2) is also applicable. Equating the expressions for $u/u_\tau$ we may evaluate $C_1$

$$C_1 = 5.1 - (0.614/aK) + (1/K)\ln(\delta u_\tau/\nu) \quad (14)$$

while from Eq. (12)

$$\Pi/K = (1/2)[(u_e/u_\tau) - 5.1 - (1/K)\ln(\delta u_\tau/\nu) + 0.614/aK] \quad (15)$$

Following procedures similar to those used by Van Driest,[4] Maise and McDonald[5] or Mathews,[2] we may obtain the profile for compressible flow

$$\frac{u}{u_e} = \frac{(B^2+4A^2)^{1/2}}{2A}\sin\{\text{arc}\sin(\frac{2A^2 - B}{(B^2+4A^2)^{1/2}}) [1 + \frac{1}{K}\frac{u_\tau}{u_e^*}(\ln\eta + \frac{2(1-\eta^a)^{1/2}}{a} - \frac{2}{a}\ln(1 + (1-\eta^a)^{1/2})) - \frac{\Pi}{K}\frac{u_\tau}{u_e^*}(2 - W(\eta))]\} + \frac{B}{2A^2} \quad (16)$$

where

$$\Pi/K = (1/2)[(u_e^*/u_\tau) - (1/K)\ln(\delta u_\tau/\nu_w) - 5.1 + 0.614/aK] \quad (17)$$

and

$$u_e^*/u_\tau = (u_e/u_\tau)(1/A)\,\text{arc}\sin[(2A^2 - B)/(B^2+4A^2)^{1/2}] \quad (18)$$

For isoenergetic flow, Eqs. (16) and (18) become, respectively,

$$\frac{u}{u_e} = \frac{1}{\sigma^{1/2}}\sin\{\text{arc}\sin\sigma^{1/2}[1 + \frac{1}{K}\frac{u_\tau}{u_e^*}(\ln\eta + \frac{2(1-\eta^a)^{1/2}}{a} - \frac{2}{a}\ln(1 + (1-\eta^a)^{1/2})) - \frac{\Pi}{K}\frac{u_\tau}{u_e^*}(1 + \cos\pi\eta)]\} \quad (19)$$

and

$$u_\tau/u_e^* = [(C_f/2)\sigma/(1-\sigma)]^{1/2}/\text{arc}\sin\sigma^{1/2} \quad (20)$$

where $2 - W(\eta)$ has been replaced by $1 + \cos\pi\eta$ for mathematical convenience.[2,5] As $a \to \infty$, Eq. (11) reduces to Eq. (1) while Eq. (19) reduces to the profile proposed by Mathews et al., Eq. (5).

The remaining problem is the selection of the constant $a$. Based on measurements reported by Klebanoff[6] and Horstman and Owen,[7] it appears that $a = 1$ represents a reasonable assumption. This amounts to the assumption of a linear shear stress distribution across the boundary layer.

The method of least squares has been used to fit the wall-wake profile, Eq. (19), for both $a = 1$ and $a \to \infty$ to a number of experimental velocity profiles reported by Seebaugh,[8] Teeter[9] and Rose.[10] An example of the results is given in Fig. 1 which shows two profiles from the study by Seebaugh of an interaction between a conical shock wave and the turbulent boundary layer at the wall of an axially symmetric $M = 2.82$ wind tunnel. The profiles are for stations just upstream and just downstream of the interaction region and the experimental velocities have been calculated from pitot pressures and the wall static pressures under the assumption of isoenergetic flow. The 10° half-angle cone used in the study did not produce a shock wave of sufficient strength to cause boundary layer separation. The values of $C_f$ and $\delta$ determined by the curve fits are listed on the figure along with values for the displacement and momentum thicknesses $\delta^*$ and $\theta$.

As is shown in the figure, both the modified wall-wake profile ($a = 1$) and the profile for $a \to \infty$ provide good representations of the experimental velocity distribution over the ranges from $y = 0$ to the values determined for $\delta$ by the curve fits. The values of $C_f$, $\delta^*$ and $\theta$ determined for the two profiles differ only slightly. However, the values of $\delta$ as determined with the modified profile show much better agreement with the values of $\delta$ based on $U/Ue = 0.995$. Furthermore, the velocity gradient for the modified profile goes to zero at $y = \delta$. In all of the data examined to date the modified profile has provided a more realistic representation of the experimental velocity distribution than the earlier version of the compressible wall wake profile.

## References


[1]Coles, D., "The Law of the Wake in the Turbulent Boundary Layer," *Journal of Fluid Mechanics,* Vol. 1, Pt. 2, 1956, pp. 191–226.

[2]Mathews, D. C., Childs, M. E., and Paynter, G. C., "Use of Coles' Universal Wake Function for Compressible Turbulent

Boundary Layers," *Journal of Aircraft*, Vol. 7, No. 2, March-April 1970, pp. 137–140.

[3]Schlichting, H., *Boundary Layer Theory*, 6th ed., McGraw-Hill, New York, 1968, p. 555.

[4]Van Driest, E. R., "Turbulent Boundary Layer in Compressible Fluids," *The Journal of the Aeronautical Sciences*, Vol. 18, 1951, pp. 145–160, 216.

[5]Maise, G. and McDonald, H., "Mixing Length and Kinematic Eddy Viscosity in a Compressible Boundary Layer," *AIAA Journal*, Vol. 6, No. 1, Jan. 1968, pp. 73–80.

[6]Klebanoff, P., "Characteristics of Turbulence in a Boundary Layer with Zero Pressure Gradient," TN 3178, 1954, NACA.

[7]Horstman, C. and Owen, F., "Turbulent Properties of a Compressible Boundary Layer," *AIAA Journal*, Vol. 10, No. 11, Nov. 1972, pp. 1418–1424.

[8]Seebaugh, W., "An Investigation of the Interaction of a Shock Wave and a Turbulent Boundary Layer in Axially-Symmetric Internal Flow Including the Effect of Mass Bleed," Ph.D. thesis, 1968, Univ. of Washington, Seattle, Wash.

[9]Teeter, G., "An Experimental Investigation of the Interaction of a Shock Wave with a Decelerated Turbulent Boundary Layer," M.S. thesis, 1971, Univ. of Washington, Seattle, Wash.

[10]Rose, W., "The Behavior of a Compressible Turbulent Boundary Layer in a Shock-Wave-Induced Adverse Pressure Gradient," Ph.D. thesis, 1972, Univ. of Washington, Seattle, Wash.

# The Influence of Pitch and Twist on Blade Vibrations

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## Nomenclature

$(EI)_{\bar{v}}$ = principal edgewise flexural rigidity, lb-ft$^2$
$(EI)_{\bar{w}}$ = principal flatwise flexural rigidity, lb-ft$^2$
$R$ = span of blade, ft
$T$ = kinetic energy, ft-lb
$U_f$ = potential energy in centrifugal force field, ft-lb
$U_S$ = strain energy in bending, ft-lb
$V$ = in-plane modal deflection function, ft
$W$ = out-of-plane modal deflection function, ft
$m$ = mass per unit length slug/ft
$m_{w,v}$ = generalized mass, slug
$k$ = generalized spring rate, lb/ft
$q$ = generalized coordinate
$r$ = spanwise position along blade, ft
$t$ = time, sec
$v$ = deflection of blade in plane of rotation, ft
$\bar{v}$ = deflection of blade in edgewise principal direction, ft
$w$ = deflection of blade normal to plane of rotation, ft
$\bar{w}$ = deflection of blade in flatwise principal direction, ft
$\Delta$ = frequency perturbation parameter, rad/sec
$\sigma$ = first mass moment of blade, slug-ft
$\theta$ = inclination of principal plane of bending, blade geometric pitch angle, rad
$\eta$ = dummy variable of integration, ft
$\omega$ = frequency of vibration, rad/sec
$\Omega$ = blade steady frequency of rotation, rad/sec
$\frac{d(\ )}{dt}$ = derivative of ( ) with respect to time
$\frac{\partial(\ )}{\partial q}$ = partial derivative of ( ) with respect to $q$
= differentiation with respect to span coordinate $r$
= differentiation with respect to time
[] = square matrix
{} = column matrix

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## Introduction

THE modes of bending vibration of rotor, prop-rotor, and propeller blades would take place in the plane of steady rotation and normal to it, if the blades were untwisted and operated at flat pitch. However the aerodynamic inflow necessitates both twist and sometimes large pitch angles, especially in the cases of the propeller and prop-rotor blades. This results in strong elastic coupling of these modes by the inclination of the blade principal planes of bending along its span. Formidable numerical complications in the vibration analysis usually result. On the other hand, a suitable energy formulation of the problem is shown to provide a simple calculation procedure employing a correction function for modifying the idealized uncoupled modes of an untwisted blade rotating at flat pitch. This approach is especially well suited for preliminary design in that it permits a rapid evaluation of the effects of the aerodynamic requirements on the structural dynamic design and behavior of these blades.

## Analysis

The Lagrangian method is employed and the analysis begins with an accounting of the kinetic and potential energies of the system. The latter is expressed in two parts: first, the strain energy in bending with respect to the two orthogonal but nonprincipal direction axes $w$ and $v$; second, the potential energy stored by virtue of the centrifugal force field. The kinetic energy of the small oscillations with respect to the steadily rotating blade is given by

$$T = \frac{1}{2}\int_0^R m(\dot{w}^2 + \dot{v}^2)dr \tag{1}$$

The potential energy of the centrifugal force field is given by[2,3,7]

$$U_F = \frac{1}{2}\Omega^2\int_0^R [\sigma(w'^2 + v'^2) - mv^2]dr \tag{2}$$

$$\sigma = \int_r^R m\eta d\eta \tag{3}$$

The potential energy due to elastic bending about the nonprincipal elastic directions is

$$U_S = \frac{1}{2}\int_0^R \left( \begin{array}{c} [(EI)_{\bar{w}} \cos^2\theta + (EI)_{\bar{v}} \sin^2\theta]w''^2 \\ + [(EI)_{\bar{w}} \sin^2 + (EI)_{\bar{v}} \cos^2\theta]v''^2 \\ +[EI)_{\bar{v}} - (EI)_{\bar{w}}] \sin 2\theta w''v'' \end{array} \right) dr \tag{4}$$

Expressing the vibratory displacements as

$$w(r, t) = q_w(t)W(r) \tag{5}$$

$$v(r, t) = q_v(t)V(r) \tag{6}$$

and the equations of motion in terms of the generalized coordinates[4] $q_w$ and $q_v$, the Lagrangian form of the equations of motion is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_w}\right) + \frac{\partial U_F}{\partial q_w} + \frac{\partial U_S}{\partial q_w} = 0 \tag{7}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_v}\right) + \frac{\partial U_F}{\partial q_v} + \frac{\partial U_S}{\partial q_v} = 0 \tag{8}$$

Writing these equations explicitly in a matrix format yields

$$\begin{bmatrix} m_w & 0 \\ 0 & m_v \end{bmatrix} \begin{Bmatrix} \ddot{q}_w \\ \ddot{q}_v \end{Bmatrix} + \begin{bmatrix} k_{ww} & k_{wv} \\ k_{vw} & k_{vv} \end{bmatrix} \begin{Bmatrix} q_w \\ q_v \end{Bmatrix} = 0 \tag{9}$$